Math 259A Lecture 18 Notes

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1 Recap Episode

1.1 New lore: examples of von Neumann algebras

From 1936 to 1943, Fred Murray and von Neumann published 4 papers titled "On Rings of Operators" I-IV. They treated the question: Are there other von Neumann factors M than $\mathcal{B}(\ell^2(I))$?

Why look at factors? Recall that since $Z(M)$ is abelian, $Z(M) \cong L^\infty(X)$ as abelian algebras for some X. If $Z(M) \neq \mathbb{C}$, and $|X|$ is finite, then $M = \bigoplus_{i=1}^n M_i$, where the M_i are factors.

Proposition 1.1. *If M is a finite dimensional C^*-algebra, then $M = \bigoplus_{i=1}^k M_{n_i \times n_i}(\mathbb{C})$.*

In general, the idea is we should have some kind of decomposition $M \cong \int M_x \, d\mu(x)$.

Theorem 1.1 (Murray-von sNeumann, 1936)**.** *If M is a factor, then either*

1. *It is type I_n (so $M \cong M_{n \times n}(\mathbb{C})$).*
2. *It is type I_∞ (so $M \cong \mathcal{B}(\ell^2(\mathbb{N}))$).*
3. *It is II_1 but not type I finite (so it is infinite-dimensional).*
4. *It is type II_∞ (so it is semifinite)*
5. *It is type III (so it has no finite projections).*

This coincided with the beginnings of ergodic theory:

Theorem 1.2 (von Neumann ergodic theorem, 1932)**.** *Let Γ be a group, and let $\Gamma \circlearrowright X$ be a measure-preserving ergodic action . Then*

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} T^n f - \int f \, d\mu \cdot 1 \right\|_2 \xrightarrow{N \to \infty} 0.$$

Many examples of II_1 factors come from these considerations of ergodic theory.

1.2 Group von Neumann algebras

Last time, we talked about group von Neumann algebras.[1]

Let Γ be a group, and let $L(\Gamma) := \overline{\operatorname{span}\lambda(\Gamma)}^{\text{wk}} = \lambda(\Gamma)'' \subseteq \mathcal{B}(\ell^2(\Gamma))$ be the group von Neumann algebra. We saw that $L(\Gamma)$ has a trace, which implies that $L(\Gamma)$ is finite. If Γ is infinite, then $L(\Gamma)$ is infinite dimensional. So to get II_1 factors, we only need a bit more. We continued this consideration by showing the following:

Theorem 1.3. *$L(\Gamma)$ is a II_1 factor if and only if Γ is ICC.*

Example 1.1. S_∞, the finite permutations of $\mathbb{N}$ is ICC.

Example 1.2. $\mathbb{F}_n$, the free group on n generators (with $n \geq 2$), is ICC.

Our proof for this theorem used intuition from Fourier analysis, which we can view as the study of $L(\mathbb{Z})$. For $\xi \in \ell^2(\Gamma)$, we considered $L_\xi : \ell^2(\Gamma) \to \ell^2(\Gamma)$ by $L_\xi(\eta) = \xi \cdot \eta$ and saw that $\|L_\xi\|_{\mathcal{B}(\ell^2,\ell^\infty)} \leq \|\xi\|_2$. So $(L_\xi, D(L_\xi))$ is a closed graph operator densely defined on $\ell^2(\Gamma)$. So if $z \in Z$, then $z(\xi_e) = \sum c_g \xi_g$. If $u_h z u_{h^{-1}} = z$ for all h, so $c_g = c_{hgh^{-1}}$ for all g, h. So z is a multiple of the identity.

[1] I don't know why this lecture is recap. But now have enough budget for the rest of the season!